EPFL

Teacher : Prof. Emmanuel Abbe
MATH-232 Probability and Statistics - Final Exam
20 June 2023
Duration : 180 minutes

359

Additional

SCIPER : 0

Do not turn the page before the start of the exam. This document is double-sided, has 24 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is **not permitted** during the exam.
- A cheat sheet is provided on the last pages of this booklet.
- For the **multiple choice** questions, we give :
  - +3 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - −1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  - +2 points if your answer is correct,
  - 0 points if you give no answer or more than one,
  - −2 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.
- The multiple choice questions are shuffled and hence are not in the order of difficultly.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren

ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer. No justifications are needed for this part.

Question 1 Suppose that random variable U follows the uniform distribution on $[0, 1]$, i.e., $U \sim \text{Uniform}(0, 1)$. Define $X = -\lambda \ln U$. What is $\mathbb{P}(X > x)$? ($\exp(z) = e^z$)

- ☐ $\frac{1}{\lambda}\exp(-x/\lambda)$
- ☐ $\exp(-x/\lambda)$
- ☐ $\lambda\exp(-\lambda x)$
- ☐ $\exp(-\lambda x)$

Question 2 Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables such that $X_n \sim \text{Poisson}(n\lambda)$ where $\lambda > 0$ is a constant. Consider the sequence given by $Y_n = \frac{X_n}{n}$ and the following claims:

(a) Y_n converges to λ in distribution.

(b) Y_n converges to λ in probability.

(c) Y_n converges to λ in mean square.

How many of the claims above are actually valid? In other words, how many modes of convergence (among in distribution, in probability and in mean square) hold for $Y_n \to \lambda$?

- ☐ 3
- ☐ 1
- ☐ 2
- ☐ 0

Question 3 Consider three bits $b_1, b_2, b_3 \in \{0, 1\}$ that are sent over a noisy channel that flips each bit independently with probability $p < \frac{1}{2}$. Assume that one transmits two possible sequences on this channel with equal probability: either 000 or 111. Therefore, we define the null and alternative hypotheses as H_0 for '000 is transmitted' and H_1 for '111 is transmitted'. What is the optimal average error probability of a hypothesis test (again, each hypothesis is selected with probability 1/2)?

- ☐ $p^2(3 - 2p)$
- ☐ $p^2(1 - p)$
- ☐ p^3
- ☐ p^2

Question 4 Consider a random variable θ taken uniformly over $[0, 2\pi]$, i.e.,

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}.$$

Note that this is equivalent with sampling a point from the unit circle randomly where θ is the angle. We define (X, Y) as the coordinates of the random point from the circle, i.e., $X = \cos\theta$ and $Y = \sin\theta$. What is $\mathbb{E}[X^2]$?
(Hint: This question is only about X but one may still use Y to come up with the answer.)

- ☐ $\frac{1}{2}$
- ☐ $\frac{3}{4}$
- ☐ $\frac{2}{3}$
- ☐ $\frac{1}{3}$

Question 5 Suppose that an individual has Covid with probability 0.01. Further, consider a Covid test that has 3% false positive and 0% false negative probability, where under H_0 'the person does not have Covid' and under H_1 'the person has Covid'. If someone tests positive, what is the probability that the person does have Covid?

- ☐ $\frac{1}{3.97}$
- ☐ $\frac{0.99}{4.96}$
- ☐ $\frac{1}{4.96}$
- ☐ $\frac{0.97}{0.99}$

Question 6 How many different ways are there to split 8 students into 4 groups of size 2?

- ☐ 2520
- ☐ 105
- ☐ 1680
- ☐ 840

Question 7 Let $X_1, X_3, X_5, X_7, \dots$ be a sequence of i.i.d. Poisson(2) random variables and define $X_{2i} = X_{2i-1}$ for $i \in \mathbb{N}$. What does $\frac{1}{n}(X_1^2 + X_2^2 + \cdots + X_n^2)$ converge to (in probability)?

- ☐ 2
- ☐ 8
- ☐ 4
- ☐ 6

Second part: true/false questions

For each question, mark the box TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8 Consider random variables $X, Y \sim \mathcal{N}_2(\begin{pmatrix}0 & 0\end{pmatrix}^T, \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix})$ having a multinomial Gaussian distribution. For a matrix $A \in \mathbb{R}^{2\times 2}$ consider the linear transformation of $\begin{pmatrix}X' \\ Y'\end{pmatrix} = A\begin{pmatrix}X \\ Y\end{pmatrix}$.
Claim: There is no other matrix other than $A = I$ (identity matrix) such that X', Y' are independent and each of them are standard normal variables, i.e., $X', Y' \sim \mathcal{N}(0, 1)$.

☐ TRUE ☐ FALSE

Question 9 A fair dice is thrown twice independently. Let D_1, D_2 be the values of the dice in the first and second throw.
Claim: the events $E_1 : D_1 = 2$ and $E_2 : D_1 + D_2 = 7$ are independent.

☐ TRUE ☐ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Unjustified answers will not get the full point even if correct; while, partial reasoning may receive some points. Leave the check-boxes empty, they are used for the grading.

Question 10: *This question is worth 5 points.*

☐ 0 ☐ .5 ☐ 1 ☐ .5 ☐ 2 ☐ .5 ☐ 3 ☐ .5 ☐ 4 ☐ .5 ■ 5

An exam has N questions which are shuffled (i.e., the order of the questions is picked uniformly at random) for each student independently. Students are taking the exam next to each other in a room. Consider a student who has 2 neighbors. What is the probability that she/he has the same question ordering as at least one of her/his two neighbors?

Question 11: *This question is worth 7 points.*

0 .5 1 .5 2 .5 3 .5
4 .5 5 .5 6 .5 7

Consider random variables X, Y following the joint normal distribution

$$X, Y \sim \mathcal{N}_2\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}\right).$$

(a) (2 points) What is the correlation between X, Y?

(b) (5 points) For a matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $b \in \mathbb{R}^2$, let

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} + b.$$

Find A, b such that X', Y' are independent and each standard normal distributed, i.e., $X' \sim \mathcal{N}(0, 1)$ and $Y' \sim \mathcal{N}(0, 1)$.

Question 13: *This question is worth 6 points.*

0 .5 1 .5 2 .5 3 .5 4 .5 5 .5 6

Let $X_1, X_2, \ldots$ be a sequence of independent geometric random variables such that $X_n \sim \text{Geometric}(\frac{\lambda}{n})$ where $\lambda > 0$ is a constant. Prove that the sequence given by $Y_n = \frac{X_n}{n}$ converges in distribution to Exponential(λ).

You may want to use the following formula for this question: $\lim_{n\to\infty}(1 - \frac{\lambda}{n})^n = e^{-\lambda}$, although there may be various ways to solve the problem.

Question 14: *This question is worth 9 points.*

0 .5 1 .5 2 .5 3 .5 4 .5
5 .5 6 .5 7 .5 8 .5 9

A referendum is going to take place in Switzerland on a specific proposal. Assume that each citizen is in favor of the proposal with probability p independently of other citizens. Answer the questions below. Note that these questions are independent of each other, so you do not need to solve them in order.

(a) (3 points) Assume $p = 0.5$. Using the Central Limit Theorem, estimate the probability that in a group of 100 citizens, at least 70 support the referendum (you should use the table provided in the cheat sheet).

(b) (3 points) For any p, consider a group of n citizens. Define $\overline{X}_n$ as the fraction of the people who support the referendum in the group. Prove that $\mathbb{P}(\overline{X}_n \geq p + \epsilon) \leq \frac{\epsilon^2}{4n}$. TYPO: Є² should be in the denominator

(c) (3 points) Assume that a polling institute wants to estimate p. To do this, the institute takes n citizens and computes $\overline{X}_n$, the fraction of people in favor of the referendum within this group. What is the smallest group size n in order to have a 95% confidence that $\overline{X}_n$ is within 0.01 of the true value of p?

Question 15: *This question is worth 8 points.*

0 .5 1 .5 2 .5 3 .5 4
.5 5 .5 6 .5 7 .5 8

Let $y_1, \ldots, y_n$ be samples drawn from the uniform distribution $U(\alpha, \beta)$ where α and β are unknown real-valued parameters.

(a) (4 points) Estimate α, β based on the method of moments.

(b) (4 points) Estimate α, β using the maximum likelihood method.

Basic formulas and definitions

- Properties of binomial coefficients
  (a) Pascal's triangle $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.
  (b) Vandermonde's formula $\sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j} = \binom{m+n}{r}$.
  (c) Negative binomial series $(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$, $|x| < 1$.
  (d) $\lim_{n\to\infty} n^{-r}\binom{n}{r} = \frac{1}{r!}$, where $r \in \mathbb{N}$ is fixed.
- Inclusion-exclusion formula:
$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1\le i_1 < i_2 < \ldots < i_r \le n} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r})$$
- Let $g(X,Y)$ be a function of a random vector (X,Y). Its conditional expectation given $X = x$ is
$$\mathbb{E}[g(X,Y)|X=x] = \begin{cases} \sum_y g(x,y) f_{Y|X}(y|x) & \text{discrete case} \\ \int g(x,y) f_{Y|X}(y|x) dy & \text{continuous case} \end{cases}$$
  on the condition that $f_X(x) > 0$ and $\mathbb{E}[|g(X,Y)||X=x] < \infty$.
- For random variables X and $Y = g(X)$, where g is a monotone increasing or decreasing function with differentiable inverse g^{-1}, we have
$$f_Y(y) = |\frac{dg^{-1}(y)}{dy}| f_X(g^{-1}(y)).$$
- For a random variable X, the moment generating function is defined as $M_X(t) = \mathbb{E}[e^{tX}]$ for $t \in \mathbb{R}$ such that $M_X(t) < \infty$. Similarly, for a random vector $X_{p\times 1} = (X_1, X_2, \ldots, X_P)^T$, we have $M_X(t) = \mathbb{E}[e^{t^T X}]$ for $t \in \mathbb{R}^P$ such that $M_X(t) < \infty$.
- The random vector $X \sim \mathcal{N}_p(\mu, \Omega)$ has a density function on $\mathbb{R}^p$ if and only if Ω is positive definite, i.e., Ω has rank p. If so, the density function is
$$f(x;\mu,\Omega) = \frac{1}{(2\pi)^{p/2}|\Omega|^{1/2}} \exp(-\frac{1}{2}(x-\mu)^T \Omega^{-1}(x-\mu)).$$
  If not, X is a linear combination of variables that have a density function on $\mathbb{R}^m$, where $m < p$ is the rank of Ω.
- Let $X \sim \mathcal{N}_p(\mu_{p\times 1}, \Omega_{p\times p})$, where $|\Omega| > 0$, and let $\mathcal{A}, \mathcal{B} \subset \{1, \ldots, p\}$ with $|\mathcal{A}| = q < p$, $|\mathcal{B}| = r < p$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Let $\mu_{\mathcal{A}}, \Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{AB}}$ be respectively the $q \times 1$ subvector of μ, $q \times q$ and $q \times r$ submatrices of Ω conformable with $\mathcal{A}, \mathcal{A}\times\mathcal{A}$ and $\mathcal{A}\times\mathcal{B}$. Then:
  - the marginal distribution of $X_{\mathcal{A}}$ is normal, $X_{\mathcal{A}} \sim \mathcal{N}_q(\mu_{\mathcal{A}}, \Omega_{\mathcal{A}})$;
  - the conditional distribution of $X_{\mathcal{A}}$ given $X_{\mathcal{B}} = x_{\mathcal{B}}$ is normal, $X_{\mathcal{A}}|X_{\mathcal{B}} = x_{\mathcal{B}} \sim \mathcal{N}_q(\mu_{\mathcal{A}} + \Omega_{\mathcal{AB}}\Omega_{\mathcal{B}}^{-1}(x_{\mathcal{B}} - \mu_{\mathcal{B}}), \Omega_{\mathcal{A}} - \Omega_{\mathcal{AB}}\Omega_{\mathcal{B}}^{-1}\Omega_{\mathcal{BA}})$.
- Let $Y = g(X) \in \mathbb{R}^n$, where $X \in \mathbb{R}^n$ is a continuous variable and
$$(X_1, \ldots, X_n) \to (Y_1 = g_1(X_1, \ldots, X_n), \ldots, Y_n = g_n(X_1, \ldots, X_n)),$$
  where g_i's are continuously differentiable. If the inverse transformation $h_i = g_i^{-1}$ exist, and we have Jacobian $J(x_1, \ldots, x_n) \in \mathbb{R}^{n\times n}$ such that $J_{ij} = \frac{\partial g_i}{\partial x_j}$ such that $|J(x_1, \ldots, x_n)| > 0$ if $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) > 0$ then
$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)|J(x_1, \ldots, x_n)|^{-1}$$
  evaluated at $x_1 = h_1(y_1, \ldots, y_n), \ldots, x_n = h_n(y_1, \ldots, y_n)$.

- Convergence of random variables: we consider the following definitions for convergence of random variables $X_1, X_2, \dots$
  - X_n converges to X in mean square, $X_n \xrightarrow{2} X$ if $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$, where $\mathbb{E}[X^2], \mathbb{E}[X_n^2] < \infty$.
  - X_n converges to X in probability, $X_n \xrightarrow{P} X$ if for all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$.
  - X_n converges to X in distribution, $X_n \xrightarrow{D} X$ if $\lim_{n\to\infty} F_n(X) = F(x)$, at each point where $F(x)$ is continuous where F represents the cumulative distribution function.
- Continuity theorem. Let $\{X_n\}, X$ be random variables with cumulative distribution functions $\{F_n\}, F$, whose MGFs $M_n(t), M(t)$ exist for $0 \le |t| < b$. If there exists a $0 < a < b$ such that $M_n(t) \to M(t)$ for $|t| \le a$ when $n \to \infty$, then $X_n \xrightarrow{D} X$.
- Combination of convergent sequences including Slutsky's Lemma. Let x_0, y_0 be constants, $X, Y, \{X_n\}, \{Y_n\}$ random variables, and h a function continuous at x_0. Then
  - $X_n \xrightarrow{D} x_0 \Longrightarrow X_n \xrightarrow{P} x_0$,
  - $X_n \xrightarrow{P} x_0 \Longrightarrow h(X_n) \xrightarrow{P} h(x_0)$,
  - $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} y_0 \Longrightarrow X_n + Y_n \xrightarrow{D} X + y_0, X_nY_n \xrightarrow{D} Xy_0$.
- Some inequalities:
  - Markov's inequality: $\mathbb{P}(X \ge a) \le \frac{\mathbb{E}[X]}{a}$ assuming that X only takes non-negative values and $a > 0$.
  - Chebyshev's inequality: $\mathbb{P}(|X - \mathbb{E}[X]| \ge a) \le \frac{\mathrm{var}(X)}{a^2}$
  - Jensen's inequality: $\mathbb{E}[g(X)] \ge g(\mathbb{E}[X])$, where g is a convex function
  - Hoeffding's inequality: Let $Z_1, \dots, Z_n$ be independent random variables with $\mathbb{E}[Z_i] = 0$ and $a_i \le Z_i \le b_i$. For $\varepsilon > 0$ and any $t > 0$, we have $\mathbb{P}(\sum_{i=1}^n Z_i \ge \varepsilon) \le e^{-t\varepsilon} \prod_{i=1}^n e^{t^2(b_i - a_i)^2/8}$. Particularly, for i.i.d. $X_1, \dots, X_n \sim \text{Bernoulli}(p)$ and $\varepsilon > 0$, we have $\mathbb{P}(|\bar{X} - p| \ge \varepsilon) \le 2e^{-2n\varepsilon^2}$, where $\bar{X} = (X_1 + \cdots + X_n)/n$.
  - Cauchy–Schwarz inequality: For random variables X, Y we have $|\mathbb{E}[XY]| \le \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$ assuming $\mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$. As a special case $\mathrm{cov}(X, Y)^2 \le \mathrm{var}(X)\mathrm{var}(Y)$ (assuming variances are defined).
- For an estimator $\hat{\theta}$ of θ we have the bias-variance decomposition $\mathrm{MSE}(\hat{\theta}) = \mathbb{E}[(\hat{\theta} - \theta)^2] = \mathrm{var}(\hat{\theta}) + b(\theta)^2$
- When we decide between the hypotheses, we can make two sorts of error:
  - Type I error (false positive): H_0 is true, but we wrongly reject it (and choose H_1);
  - Type II error (false negative): H_1 is true, but we wrongly accept H_0.

		Decision	
		Accept H_0	Reject H_0
State of nature	H_0 true	Correct choice (true negative)	Type I error (false positive)
	H_1 true	Type II error (false negative)	Correct choice (true positive)

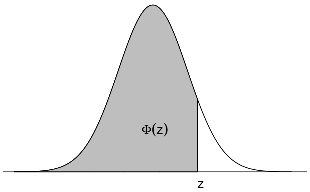
  Further, we call
  - the false positive probability the *size* α of the test, and
  - the true positive probability the *power* β of the test.
- Pearson statistic (or chi-square statistic). Let $O_1, \dots, O_k$ be the number of observations of a random sample of size $n = n_1 + \cdots + n_k$ falling into the categories $1, \dots, k$, whose expected numbers are $E_1, \dots, E_k$,where $E_i > 0$. Then the Pearson statistic (or chi-square statistic) is $T = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i}$. If the joint distribution of $O_1, \dots, O_k$ is multinomial with denominator n and probabilities $p_1 = \frac{E_1}{n}, \dots, p_k = \frac{E_k}{n}$, then $T \dot{\sim} \chi^2_{k-1}$, the approximation being good if $k^{-1} \sum E_i \ge 5$.

- Neyman–Pearson Lemma. Let $f_0(y), f_1(y)$ be the densities of Y under simple null and alternative hypotheses. Assume the set $\mathcal{Y}_\alpha = \{y \in \Omega : \frac{f_1(y)}{f_0(y)} > t_\alpha\}$ such that $\mathbb{P}_0(Y \in \mathcal{Y}_\alpha) = \alpha$ exists. Then, $\mathcal{Y}_\alpha$ maximises $\mathbb{P}_1(Y \in \mathcal{Y}_\alpha)$ amongst all the $\mathcal{Y}'$ such that $\mathbb{P}_0(Y \in \mathcal{Y}') \leq \alpha$. Thus, to maximise the power of a given threshold, we must base the decision on $\mathcal{Y}_\alpha$ ($\mathcal{Y}_\alpha$ should be the reject region for H_0).

Distributions

Distribution	PMF/PDF	Expected Value	Variance	MGF
Bernoulli Bern(p)	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$	$1-p+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$ $k\in\{0,1,2,\dots n\}$	np	$np(1-p)$	$(1-p+pe^t)^n$
Geometric Geom(p)	$P(X=k)=(1-p)^{k-1}p$ $k\in\{1,\,2,\,\dots\}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$ $(1-p)e^t<1$
Neg. Binom. NegBin(r,p)	$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x\in\{r,r+1,r+2,\dots\}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$(\frac{pe^t}{1-(1-p)e^t})^r$ $(1-p)e^t<1$
Hypergeom. HypG(w,b,n)	$P(X=k)=\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$ $k\in\{0,1,2,\dots,n\}$	$\mu=\frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$	messy
Poisson Pois(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k\in\{0,\,1,\,2,\,\dots\}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform U(a,b)	$f(x)=\frac{1}{b-a}$ $x\in[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential exp(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in(0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in(-\infty,\infty)$	μ	σ^2	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
Chi-Square χ^2_n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x\in(0,\infty)$	n	$2n$	$(1-2t)^{-n/2}$ $t<1/2$

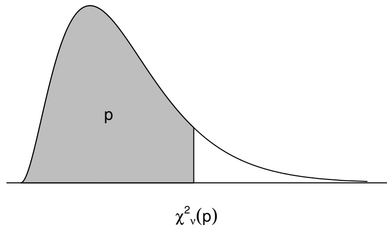
Standard normal distribution $\mathbf{\Phi(z)}$



For $z < 0$ we use symmetry: $\mathbb{P}(Z \leq z) = \Phi(z) = 1 - \Phi(-z)$, $z \in \mathbb{R}$.

z	0	1	2	3	4	5	6	7	8	9
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56750	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84850	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92786	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

χ^2_ν distribution



$\chi^2_\nu(p)$: quantiles for the chi-square distribution with ν degrees of freedom.

ν	.005	.01	.025	.05	.10	.25	.50	.75	.90	.95	.975	.99	.995	.999
1	0	.0002	.010	.0039	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6	13.8
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	20.7	22.2	24.4	26.5	29.1	33.7	39.3	45.6	51.8	55.8	59.3	63.7	66.8	73.4
50	28.0	29.7	32.4	34.8	37.7	42.9	49.3	56.3	63.2	67.5	71.4	76.2	79.5	86.7
60	35.5	37.5	40.5	43.2	46.5	52.3	59.3	67.0	74.4	79.1	83.3	88.4	92.0	99.6
70	43.3	45.4	48.8	51.7	55.3	61.7	69.3	77.6	85.5	90.5	95.0	100.	104.	112.